D-MAGIC LABELINGS OF GRAPHS

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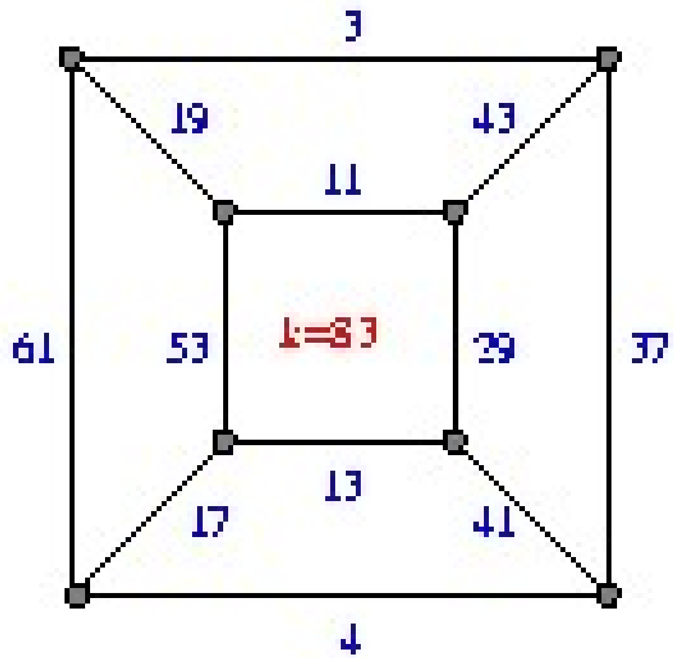
Magic Labeling: The Beginning

Definition Sedláček (1963)

A magic labeling is a one-to-one mapping $f: E \to \mathbb{R}^+$ with the property that there is a constant k such that at any vertex x

$$\sum_{y \in N(x)} f(xy) = k$$

where $N(x)$ is the set of vertices adjacent to x.

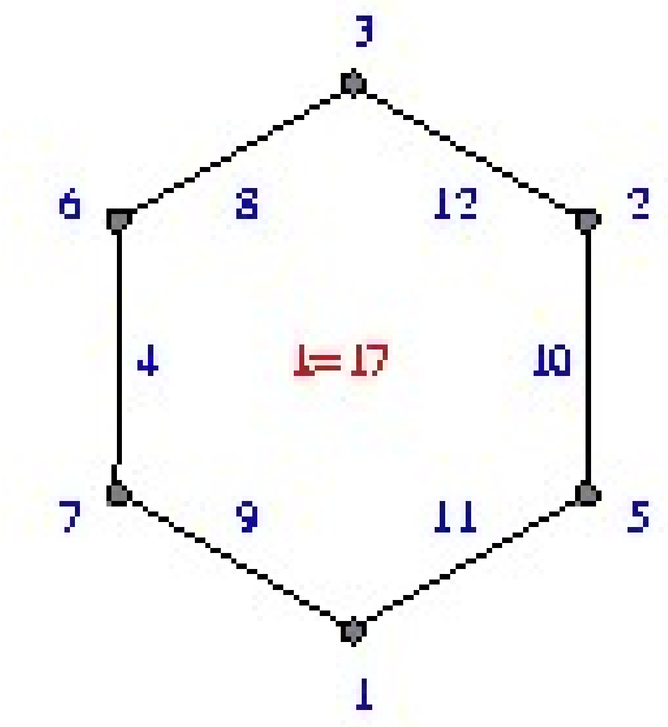


Magic Labeling: The Beginning

Definition Kotzig and Rosa (1970)

An edge-magic total labeling is a bijection $f: V \cup E \rightarrow \{1,2,\ldots,|V \cup E|\}$ with the property that there is a constant k such that at any edge xy,

$$f(x) + f(xy) + f(y) = k.$$



Magic Labeling: Open Problem and Conjecture

Not all graphs are edge-magic.

The edge-magic property is not monotone with respect to the subgraph relation.

Conjecture Kotzig & Rosa (1970)

All trees are edge-magic total.

Question Erdős (Kalamazoo 1996)

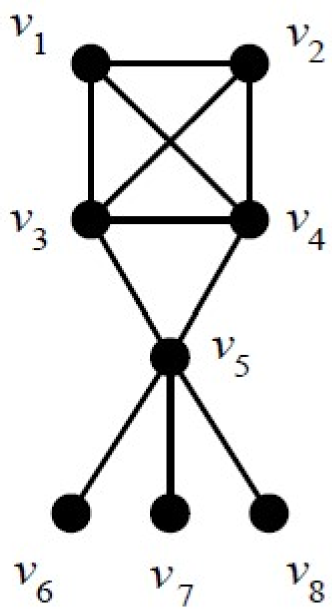
What is the maximum number of edges in an edge-magic graph?

$\frac{2}{7}n^2 + O(n) \leq \mathcal{M}(n) \leq (0.489 \ldots + o(1))n^2$ Pikhurko (2006)

Distance in Graph

The distance of two vertices *u* and *v* is the length shortest path connecting *u* and *v*.

The greatest distance between two vertices i



e diameter of *G*.

Distance Magic Labeling

A natural extension of Kotzig & Rosa's magic labeling.

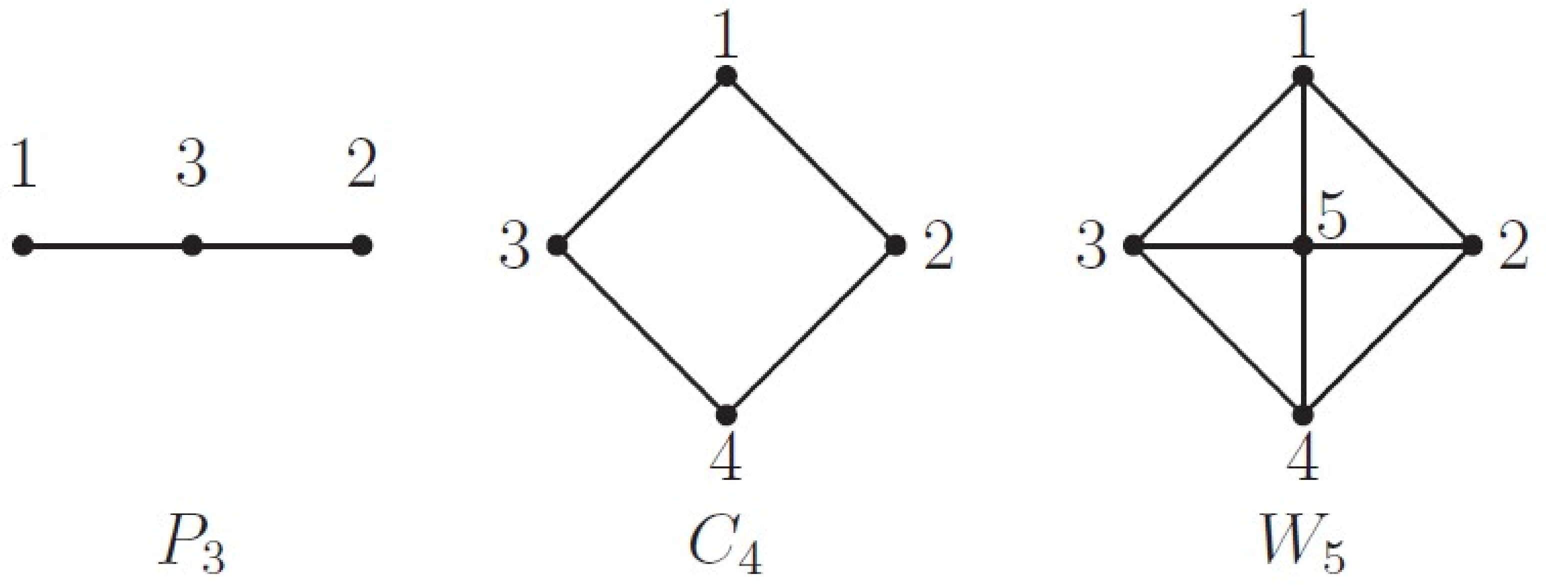
Definition Vilfred (1994)

A distance magic labeling is a bijection $f: V \rightarrow \{1, 2, \ldots, |V|\}$ with the property that there is a magic constant k such that at any vertex x, the weight

$$w(x) = \sum_{y \in N(x)} f(y) = k$$

where $N(x)$ is the set of vertices adjacent to x.

Smallest Distance Magic Graphs



Application

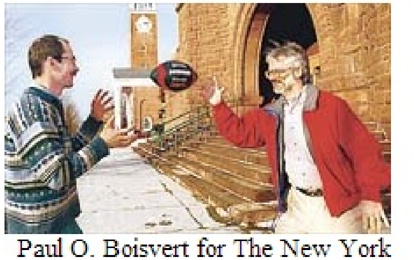
Designing incomplete tournaments

- a fair incomplete tournament,
- an equalized incomplete tournament, or
- a handicap incomplete tournament

Mathematicians Solve XFL's Scheduling Problem

By PATRICIA COHEN

Here's the scheduling problem that faced the XFL, the new smack-in-the-mouth, kick-in-the-groin professional football league that has its premiere tonight.



Paul O. Boisvert for The New York Times
Dalibor Froncek, left, and Jeff Dinitz, experts in combinatorics.

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Two divisions with four teams each; each team plays every other team in its own division twice and the teams in the other division once. So far so good. Then the headaches began. Marquee name games � like the Las Vegas Outlaws against the New York/New Jersey Hitmen � had to take place on Saturday nights to mesh with NBC's schedule, and the openers had to be in warm- weather locations (no blizzards) and smaller stadiums (so the stands would

New York Times, 3 Februa

Equalized Incomplete Tournament

A fair way to schedule a tournament is to create a round-robin, where each team plays every other team once. However, playing that many games is not always possible. Thus, we need a fair way of scheduling when not all of the games from the round-robin can be played.

In order to be fair, each team should play the same number of teams and the difficulty of the schedule for each team should mimic the difficulty of playing the entire round-robin tournament.

To assist with considering the difficulty of each team's schedule, rank the teams from strongest to weakest so that the strongest team has rank 1.

Define the strength of the i-th ranked team as $s_n(i) = n + 1 - i$.

An equalized incomplete tournament of n teams with r rounds, $EIT(n,r)$, is a tournament in which every team

Equalized Incomplete Tournament of 6 Rounds for 12 Teams

Team	Round 1	Round 2	Round 3	Round 4	Round 5	Round 6
1	2	4	6	7	9	11
2	1	3	5	8	10	12
3	4	2	7	9	11	6
4	3	1	8	10	12	5
5	9	6	2	11	7	4
6	10	5	1	12	8	3
7	12	10	3	1	5	8
8	11	9	4	2	6	7
9	5	8	12	3	1	10
10	6	7	11	4	2	9
11	8	12	10	5	3	1
12	7	11	9	6	4	2

Distance Magic Regular Complete Multipartite Graphs

Theorem Vilfred (1994) Miller, Rodger, RS (2003)

The complete symmetric multipartite graph $H_{n,p}$ with p parts and n vertices in each part , has a distance magic

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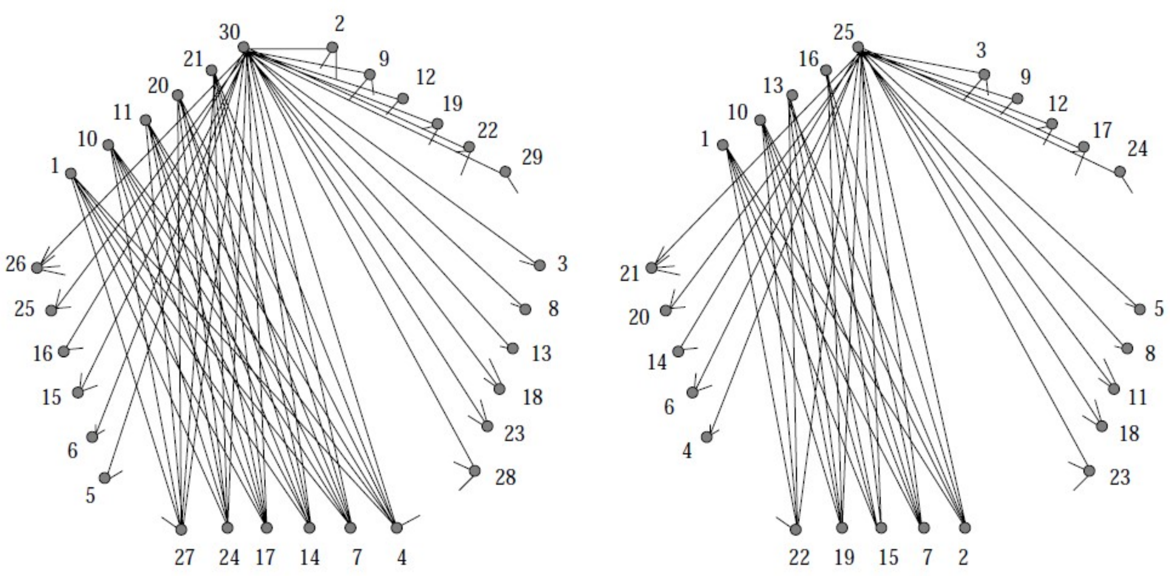


$H_{6,5}$ $H_{5,5}$

Distance Magic Complete Multipartite Graphs

Conjecture Miller, Rodger, RS (2003)

Let $1 \le a_1 \le \cdots \le a_p, p > 1$. Let $s_i = \sum_{j=1}^{i} a_j$ and $n = s_p$. There exists a distance magic labeling of the complete multipartite graph $K_{a_1, a_2, \cdots, a_p}$ if and only if the following conditions hold.

a) $a_2 \ge 2$.

b) $n(n+1) \equiv 0 \bmod 2p$, and

c) $\sum_{j=1}^{s_i}(n+1-j) \ge \frac{in(n+1)}{2p}$ for $1 \le i \le p$.

True for

p=2 Vilfred (1994) Miller, Rodger, RS (2003) Beena (2009)

p=3 Miller, Rodger, RS (2003)

p=4 Kotlar (2016)

with Chris Rodger (Auburn) and Mirka Miller † (Newca

Magic Constant

Let G be a nontrivial distance magic graph of order n with distance magic labeling f and magic constant k.

Observation $k \geq n$.

Theorem Vilfred (1994)

$k = n$ if and only if $G \approx P_3$ or $G \approx P_3 \cup tC_4$.

Theorem Jinnah (1999)

$k = n + 1$ if and only if $G \approx tC_4$.

Theorem Arumugam, Kamatchi, Kovar (2016)

There is no distance magic graph with $k = n + 2$.

For any odd integer $k \geq 3$, there exists a

Embedding

Vilfred (1994)

Every graph is a subgraph of a distance magic graph

Acharya, Rao, Singh & Parameswaran (2004)

Every graph is an induced subgraph of a regular distance magic graph

Rao, Singh & Parameswaran (2004)

Every graph H is an induced subgraph of an Eulerian distance magic graph G where the chromatic number of H is the same as G

There is no forbidden subgraph characterization for distance magic graphs

Distance Magic Graphs are Rare

Catalogue of distance magic graphs up to 9 vertices. Yasin & RS (2015)

n	# non-isomorphic graphs	# non-isomorphic distance magic graphs
1	1	1
2	2	1
3	4	2
4	11	2
5	34	2
6	156	2
7	1044	4
8	12346	6
9	275668	6

Generalization: D-Magic Labeling

Definition O'Neal & Slater (2011)

Let d be the diameter of a graph G and $D \subseteq \{0,1,2, \dots, d\}$ be a set of distances in graph G.

A bijection $f : V \rightarrow \{1, 2, \dots, |V|\}$ is said to be a D-magic labeling if there exists a D-magic constant k such that for any vertex x, the weight

$w(x) = \sum_{y \in N_D(x)} f(y) = k$, where $N_D(x) = \{y \in V \mid d(x, y) \in D\}$.

A graph admitting a D-magic labeling is called D-magic.

If $D = \{1\}$, a D-magic labeling is known as a distance magic labeling. Vilfred (1994)

If $D = \{0,1\}$, a D-magic labeling is called a closed distance magic labeling. Beena (2009)

Two Questions

Let G be a graph of diameter d.

1. Is D-magic constant of G unique?

2. Does there always exist a non-empty distance set $D \subset \{0,1,2,\dots,d\}$ such that G is D-magic?

The D-Magic Constant is Unique

Definition O'Neal & Slater (2013)

A function $g : V \rightarrow [0, 1]$ is said to be a D-neighborhood fractional dominating function if for every vertex v, $\sum_{u \in N_D(v)} g(v) \geq 1$.

The D-neighborhood fractional domination number of a graph is denoted by $\gamma_{ft}(G; D)$ and is defined as $\min\left\{\sum_{v \in V(G)} g(v) | g \text{ is a } D\text{–neigborhood fractional dominating function}\right\}$.

Theorem O'Neal & Slater (2013)

If a graph G is D-magic, then its magic constant is

$$k = \frac{n(n+1)}{2\gamma_{ft}(G; D)}$$

Two Questions

Let G be a graph of diameter d.

1. Is D-magic constant of G unique? YES

2. Does there always exist a non-empty distance set $D \subset \{0,1,2,\ldots,d\}$ such that G is D-magic?

D-Magic Strongly-Regular Graphs

A graph G is strongly-regular with parameter (r, a, c) if G is r-regular, every adjacent pair of vertices has a common neighbors, and every nonadjacent pair has c common neighbors.

Theorem Anholcer, S. Cichacz, and I. Peterin (2016)

RS and Anuwiksa (2019+)

Let G be a strongly-regular graph.

G is D-magic if and only if

1) G is the complete multipartite graph $H_{p,m}$, where p is even or both p and m are odd, for $D = \{1\}$ or $D = \{0,2\}$,
2) G is the complete graph K_n, for $D = \{0,1\}$ or $D = \{2\}$.

Two Questions

Let G be a graph of diameter d.

1. Is D-magic constant of G unique? YES

2. Does there always exist a non-empty distance set $D \subset \{0,1,2,\ldots,d\}$ such that G is D-magic? NO

Spectrum of Regular Distance Magic Graphs

Let G be a distance magic graph with $V(G) = \{v_1, v_2, \dots, v_n\}$ and f a distance magic labeling of G.

Suppose that $w(v_i) = \sum_{x \in N(v_i)} f(x)$. Then $w(v_i) = k$ for each i.

This can be rewritten as

$$A\vec{l} = k\vec{u}_n$$

where A is the adjacency matrix of G,

$\vec{l} = (f(x_1), f(x_2), \dots, f(x_n))^t$ and $\vec{u}_n = (1, 1, \dots, 1)^t$.

If G is r-regular, then $A\vec{l}_1 = k\vec{u}_n$ with $\vec{l}_1 = \left(\frac{k}{r}, \frac{k}{r}, \dots, \frac{k}{r}\right)^t$.

Thus $A(\vec{l} - \vec{l}_1) = 0$ and so $0 \in Sp(G)$.

with Aholiab Tritama (Deakin), Yeva Ashari (ITB), and Pal

Two Questions

Let G be a graph of diameter d.

1. Is D-magic constant of G unique? YES

2. Does there always exist a non-empty distance set $D \subset \{0,1,2,\ldots,d\}$ such that G is D-magic? NO

D-Magic Hypercubes

The hypercube on n vertices, denoted by Q_n, is the graph whose vertices are the binary n-vector and two vertices are adjacent iff the corresponding vectors differ in exactly one coordinate.

Question

Does there exist a non-empty distance set $D \subset \{0,1,2,\ldots,n\}$ such that the hypercube Q_n is D-magic?

Theorem Gregor and Kovář (2013) Cichacz, Froncek, Krop, Raridan (2016)

The hypercube Q_n is $\{1\}$-magic if and only if $n \equiv 2(mod 4)$.

D-Magic Labelings for Q_n, $n \equiv 2(mod 4)$

Theorem Anuwiksa, Munemasa, and RS (2019+)

If $n \equiv 2(mod 4)$ then there exists a D-magic labeling of the hypercube Q_n whenever D is of the form

$$E \cup \bigcup_{i \in I} \{i, n-i), $$

where $E \subseteq \{1,3,5, \dots\}, I \subseteq \{0,1, \dots, \frac{n}{2}\}$, and $E \cap \{i, n-i\} = \emptyset$ $(i \in I)$.

Open Problem

Are these the only Ds for which the hypercube Q_n, $n \equiv 2(mod 4)$, is D-magic?

D-Magic Labelings for Q_n, $n \equiv 1(mod4)$

Theorem Anuwiksa, Munemasa, and RS (2019+)

The hypercube Q_n is $\{0,1\}$-magic if and only if $n \equiv 1(mod4)$.

Theorem

Let $n \equiv 1(mod4)$. The hypercube Q_n is D-distance magic, whenever D is of the form

$$\bigcup_{i \in I_1} \{2i, 2i+1\} \cup \bigcup_{j \in I_2} \{j, n-j\},$$

where $I_1, I_2 \subseteq \{0,1, \dots, \frac{n-1}{2}\}$ and $\{2i, 2i+1\} \cap \{j, n-j\} = \emptyset$ $(i \in I_1, j \in I_2)$.

Open Problem

Are these the only Ds for which the hypercube Q_n, $n \equiv 1(mod4)$, is D-magic?

Question

For $n \equiv 0{,}3(mod 4)$, does there exist a non-empty distance set $D \subset \{0,1,2,\ldots,n\}$ such that the hypercube Q_n is D-magic?

with Hajime Tanaka (Tohoku) and Akihiro Munemasa (Tol

Another Generalization: Γ −Distance Magic Labeling

Definition Froncek (2013)

Let G be a graph on n vertices and Γ be an Abelian group of order n.

A bijection $f : V \rightarrow \Gamma$ is said to be a Γ −distance magic labeling if there exists a magic constant k such that for any vertex $x, w(x) = \sum_{y \in N(x)} f(y) = k$.

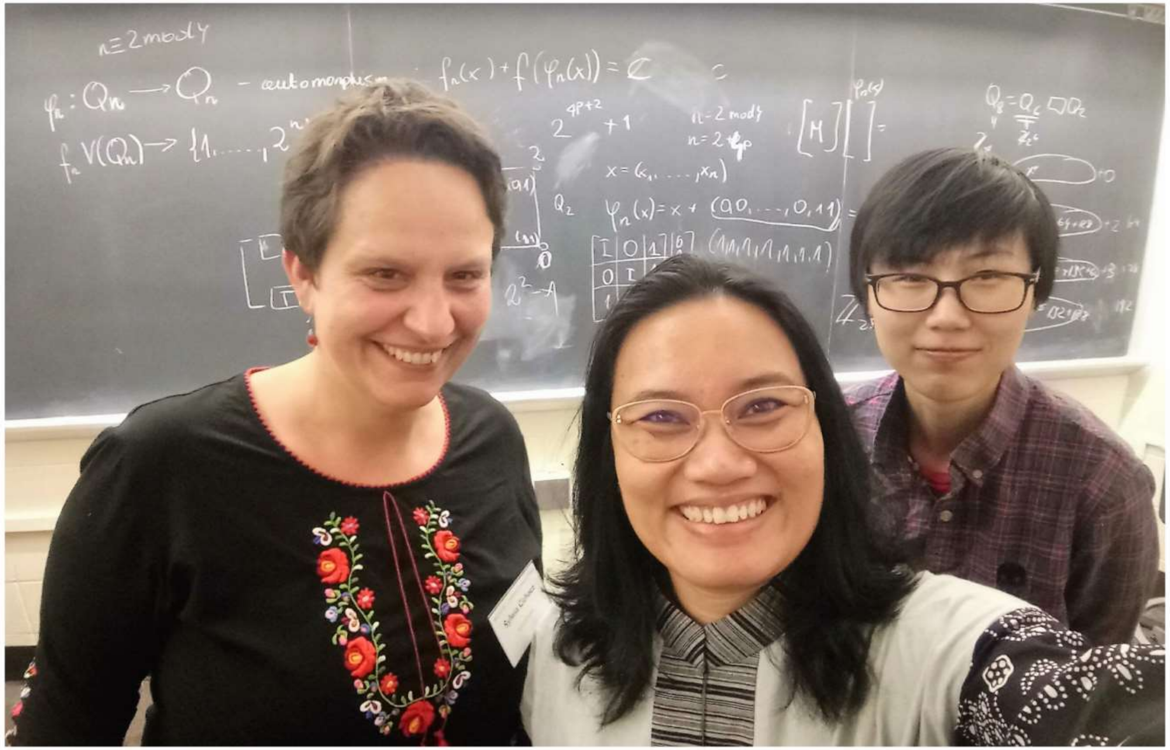
A graph G is called group distance magic if there exists a Γ −distance magic labeling for every Abelian group Γ of order n.

Group Distance Magic Hypercubes

Theorem Cichacz, Froncek, RS, Qiu (2019+)

The hypercube Q_n is group distance magic if and only if n is even.

The hypercube Q_n is group closed distance magic if and only if n is odd.



Duluth, 6 October 2018

with Jiangyi Qiu (UMass Amherst), Sylwia Cichacz
(AGH Krakow), and Dalibor Froncek (UM Duluth)

References

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